# *Studies of Dual Harmonic Acceleration in ISIS*


## C R Prior

*Rutherford Appleton Laboratory, Chilton, Didcot, Oxon*



Studies have been made of the addition of a harmonic number, $h = 4$, radio-frequency system to the existing $h = 2$ system in ISIS. The design aim is an improvement in the beam bunching factor for the acceleration of more intense beams. Use has been made of a longitudinal space-charge tracking code to define a set of parameters for the acceleration of $3 \times 10^{13}$ protons per pulse, 240μA average. This corresponds to an increase of 20% in the maximum intensity achieved to date. Further studies indicate that it may in fact be possible to accelerate $3.7 \times 10^{13}$ protons per pulse, 295μA, by these means.


### *Equations of Particle Motion*

The motion of particles in a synchrotron is normally studied through the equations

$$\frac{d}{dt}\frac{\Delta E}{\omega_0} = \frac{e}{2\pi}\left[V(\phi) - V(\phi_s) + V_{sc}(\phi)\right] \tag{1}$$

$$\frac{d}{dt}\Delta\phi = \frac{h\omega_0\eta}{\beta^2 E_s}\Delta E \tag{2}$$

where $\Delta E = E - E_s$; $\eta = 1/\gamma_t^2 - 1/\gamma^2$; $E$ and $\phi$ are the energy and phase of the particle; $V(\phi)$ is the accelerating waveform; and $h$ is the harmonic number. The suffix $s$ refers to the synchronous particle and, since the equations govern only the first-order variations from synchronous values, the revolution frequency $\omega_0$ is the value for the synchronous particle to a first approximation. $V_{sc}$ represents the space-charge forces generated within the beam and is given by

$$V_{sc}(\phi) = -e\frac{d\lambda(\phi)}{d\phi}\left[\frac{Rg_0}{2\varepsilon_0\gamma^2} - L\beta^2c^2\right] \tag{3}$$

where $\lambda$ is the line density of particles, $L$ is the total inductance per turn of the reactive wall, $R$ the radius of the machine, and, for a beam with circular cross-section of mean radius $a$ in a circular pipe of radius $b$, $g_0 = 1 + 2\ln(b/a)$.

### *Applied Voltage*

When the applied voltage is sinusoidal, $V(\phi) = \hat{V}\sin\phi$, the behaviour of the particle beam, either in the absence of space-charge or in the case of simple particle distributions, is well known.[1,2] Here we wish to study the effect of introducing an additional acceleration component of harmonic number $2h$, giving a combined voltage which may be written in the form[3]

$$V(\phi) = \hat{V}(t)[\sin(\phi) - \delta\sin(2\phi + \theta)] \tag{4}$$

The problem is to find a continuous sequence of parameters, $\delta$ and $\theta$, to find a stable system optimised in the sense of being able to contain intense beams with good bunching factor and low loss.

By balancing the energy gain to the effect of the guide magnetic field, we can find the synchronous phase through the equation

$$2\pi R\rho\dot{B} = \hat{V}[\sin\phi_s - \delta\sin(2\phi_s + \theta)]. \tag{5}$$

where $\rho$ is the mean bending radius. For convenience, we shall denote by $A$ the acceleration term $2\pi R\rho\dot{B}/\hat{V}$. When $\delta = 0$, assuming $A < 1$, there are two solutions of equation (4) in the range $[-\pi, \pi]$. One, $\sin^{-1}A$, defines the phase of the synchronous particle and the other, $\pi - \sin^{-1}A$, gives the maximum limit of stability. An example of the voltage shape when $\delta \neq 0$ is shown in Figure 1, which is drawn for

$\delta = 0.58$ and $\theta = -60°$. This suggests that, for a limited range of values of A, it may be possible for four solutions to exist.

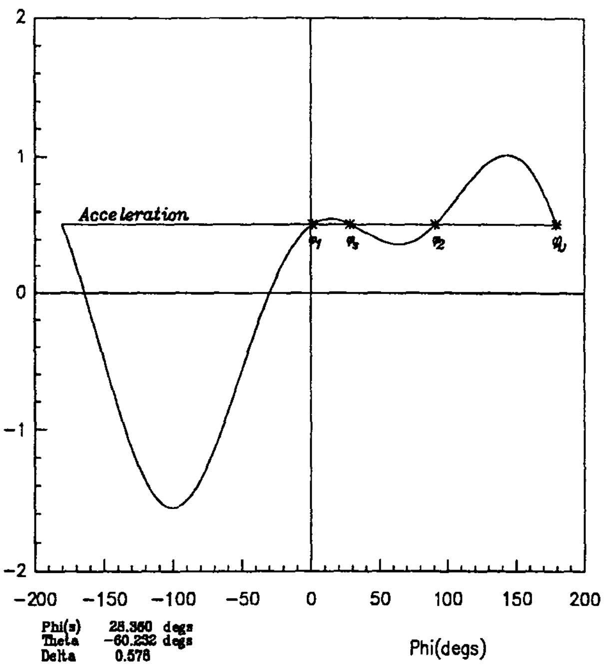


Figure 1. Shape of the Dual Harmonic Voltage Function

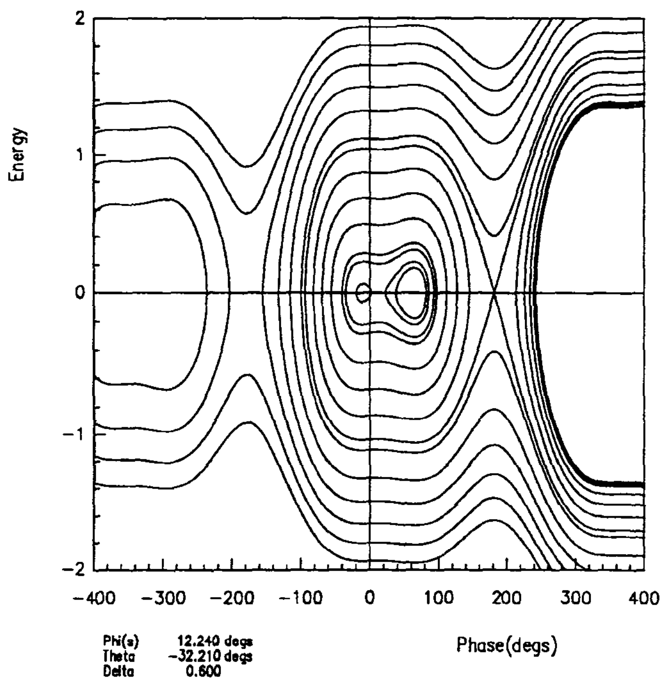


Figure 2. Phase-space Trajectories

The nature of these solutions may be found by looking at the the phase-space trajectories of particles, given by constant values of the Hamiltonian. Ignoring space-charge,

$$H = \frac{h\eta}{2\beta^2 E_s}(\Delta E)^2 - \frac{e}{2\pi}U(\phi) \tag{6}$$

where

$$U(\phi) = \int_{\phi_s}^{\phi} [V(\phi') - V(\phi_s)]d\phi'$$
$$= -\hat{V}[\cos\phi - \cos\phi_s - \tfrac{1}{2}\delta\{\cos(2\phi + \theta) - \cos(2\phi_s + \theta)\} + A(\phi - \phi_s)]$$

A typical plot, such as that shown in Figure 2, reveals that two of the points, $\phi_1$ and $\phi_2$, define stable synchronous particles. The solution $\phi_v$ defines the extreme unstable fixed point and the fourth represents a locally unstable fixed point near the centre of the stable region. It is this fourth solution that we take as $\phi_s$.

Equation (5) is equivalent to the condition $dU/d\phi = 0$ and comparison of graphs of $U(\phi)$ for $\delta = 0$ and $\delta > 0$ (Figure 3) make clear that the introduction of the extra minimum leads to an extended region of $\phi$ over which stable oscillations are possible.

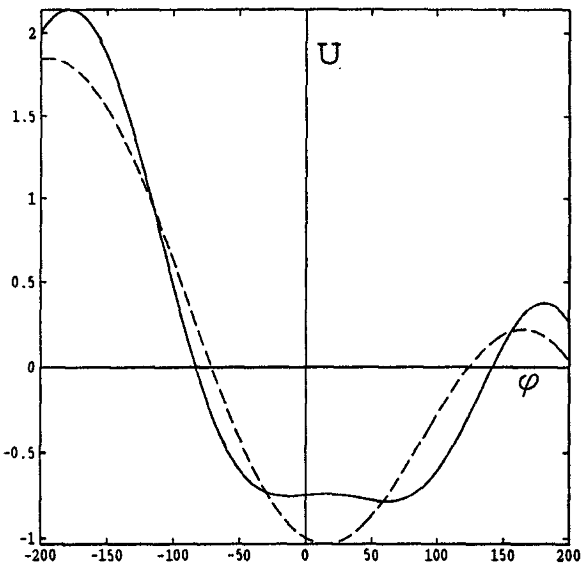


Figure 3. Potential Energy for $\delta > 0$ (continuous curve) and $\delta = 0$ (dashed curve).

Bunching factor, which is important since it is directly proportional to the maximum amount of beam that can be contained by the system, is also enhanced. Analytically this may be demonstrated by looking at the Hofmann-Pedersen particle distribution[2], which is a self-consistent model in which the space charge-forces have the same form as the applied voltages. For such a beam, the line density may be written

$$\lambda(\phi) = N_b \frac{U(\phi_v) - U(\phi)}{u_L}$$

where $N_b$ is the number of particles in the bunch and

$$u_L = \int_{\phi_u}^{\phi_v} [U(\phi_v) - U(\phi)]d\phi$$

The bunching factor is then $N_b/2\pi\lambda(\hat{\phi})$, with $\hat{\phi}$ the value of $\phi$ corresponding to the peak value of $\lambda$ (either $\phi_1$ or $\phi_2$). Defining

$$f(\phi_u, \phi_v) = \sin\phi_v - \sin\phi_u - \tfrac{1}{2}(\phi_v - \phi_u)(\cos\phi_v + \cos\phi_u)$$

this can be written most simply as

$$B_f = \frac{1}{2\pi} \frac{f(\phi_u, \phi_v) - \tfrac{1}{4}\delta f(2\phi_u + \theta, 2\phi_v + \theta)}{U(\hat{\phi}) - U(\phi_v)}. \qquad (7)$$

These equations may be solved numerically and show that a non-zero $\delta$ can produce a bunching factor increased by as much as 50%. For example, when $\delta = 0$, $\phi_s = 28°$, $B_f = 0.29$, but with $\delta = 0.58$, $\phi_s = 28°$ and $\theta = -60°$, $B_f = 0.43$, an increase of 49.1%.

As a rough guide to the values of $\theta$ to use, we may note that $\theta = -2\phi_s$ will leave the synchronous phase unaltered when the additional voltage terms are introduced, but a more detailed analysis is needed to find the best scheme for the complete bunching process in a synchrotron. The quantity $A$ for example depends critically on the ratio $\dot{B}/\hat{V}$ and, if the cavity voltages are not high enough to keep $A$ less than about 0.7 it turns out that it is not possible to find a combination of $\theta$ and $\delta$ to give four solutions of (5). In addition, even when there are four solutions, the stable region can collapse at $\phi_s$ and degenerate into two separate, small, regions centred on $\phi_1$ and $\phi_2$. To investigate these effects in detail requires numerical analysis of equations (5) and (6). This reveals that for values of $A \leq 0.5$ most values of $\delta \leq 0.6$ and $\theta \leq 0.0$ give enhanced stable regions of phase-space, but as soon as $A$ increases above 0.5, the range of acceptable $\delta$ and $\theta$ diminishes, and above 0.65 only a very limited range of $\delta$ and $\theta$ will work. The conclusion is that a suitable dual harmonic system can only be devised if the design voltages of the cavities are high enough to prevent A becoming too large, thereby allowing a complete set of continuous $\delta$- and $\theta$-values for the whole cycle. Given this, the condition for non-degeneracy of the stable region is

$$\eta U(\phi_v) > 0$$

where $\phi_v$ is the maximum phase excursion. For continuity under acceleration as $A$ increases from zero (starting at time $t = 0$), we require

$$\phi_s = 0 \quad \theta = 0 \quad \delta = 0.5 \quad \text{at } t = 0. \tag{8}$$

The three roots $\phi_1$, $\phi_s$, $\phi_2$ then coincide at the start of acceleration and immediately start to separate. For maximum area of the stable region, it is also found that $\theta$, while always negative, should have as small an absolute value as possible.

## *Simulation Model for ISIS*

The behaviour of the beam in ISIS from the start of injection through trapping and acceleration to an energy of $800 MeV$, a timespan of just over 10msec, has been modelled by means of a one-dimensional tracking code based on the equations of motion (1) and (2). The injected beam has been assumed to be uniformly spread in phase from $-\pi$ to $\pi$ and to have parabolic momentum spread corresponding to $\Delta p/p = \pm 2 \times 10^{-3}$. The simulation uses less than 20,000 particles in total but avoids the problem of having only a few unrepresentative particles for each injected turn by allowing particles to carry different charges and superimposing incoming beam onto existing beam by a weighted method of charge allocation onto an imaginary grid. Space charge is calculated from the line density according to equation (3) after a measure of smoothing has been carried out to remove statistical effects. Other parameters used in the model are

$$h = 2 \quad R = 26\text{m} \quad \rho = 7\text{m} \quad \gamma_t = 5.05 \quad g_0 = 1.75$$

$$B(t) = B_0 - B_1 \cos(2\pi f t) \tag{9}$$

where

$$B_0 = 0.4369T \quad B_1 = 0.2604T \quad f = 50Hz.$$

The first simulation runs were made with a beam of $2 \times 10^{13}$ protons per pulse and a simple sinusoidal voltage ($\delta = 0$). These were used to determine the optimum voltage profile for $\hat{V}$ and also determine the point in the process at which the voltage should be switched on, so as to balance the requirements of low loss and small momentum spread. The result was a voltage profile looking broadly like that shown in Figure 4 but scaled to a peak of $140kV$. Loss was predicted to be under 5%, all occurring in the first 1.5msec of the cycle, and was found to be minimised by switching on the r.f. cavities as early in the process as possible. Maximum momentum spread was less that $7.7 \times 10^{-3}$.

For the dual harmonic model, the definition of $A$, equations (5) and (9) give the restriction

$$A_{max}\hat{V}_{max} \geq 2\pi R \rho \dot{B}\,|_{max} = 93.54\ kV \tag{10}$$

If the maximum cavity voltage is $140kV$, $A$ will exceed 0.67 and at this level it is at best extremely difficult to design a stable system. The full design capacity of the cavities must therefore be used and the voltage profile re-scaled to a peak of $160kV$ ($A \leq 0.6$) as shown in Figure 4 and the first two columns of Table 1.

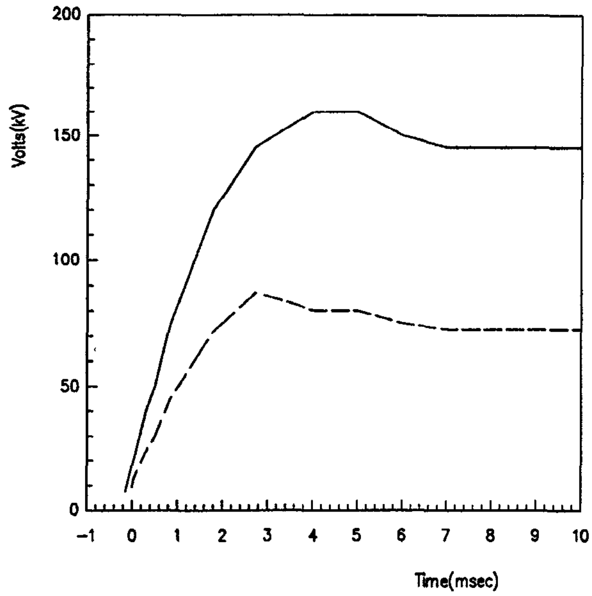


Figure 4. Cavity Voltages for ISIS Simulation Model
(The solid curve represents $\hat{V}(t)$; the dashed curve shows
the second harmonic voltage $\delta\hat{V}$.)

A suitable set of values of θ and δ corresponding to the voltages can then be generated numerically. A fully optimised system (based on maximising the bunching factor) would not be practical since the parameters vary too rapidly and too widely to make economic sense, but we can follow the method used to control the cavity voltages. In ISIS these are prescribed as a set of values at fixed times between which the controlling computer carries out linear interpolation in the form of very small step functions. Similar schemes have been produced for δ and θ, keeping to the voltage data times where possible, and these are shown in Figures 5 and 6, with specific values laid out in Table 1.

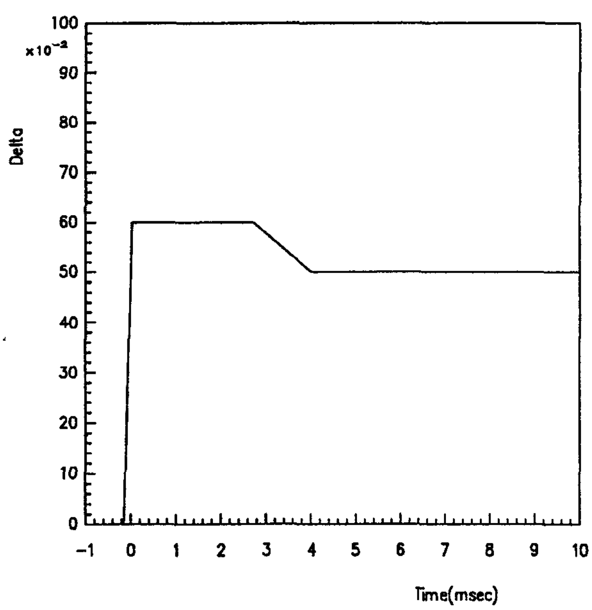


Figure 5. δ-ratio of Cavity Voltages.

With this model, δ is switched on at $t = -0.15$msec and rises through 0.5 at $t = 0.0$msec (equation (8)) to a maximum of 0.6. This peak value, determined by design and cost limitations, is maintained for just under 3msec, but, because we also require that the $2h$-harmonic voltage does not exceed $80kV$, $\delta$ is brought down to 0.5 at 4msec and held at this level through to the end of the cycle. This maintains the extended bucket and allows the three fixed points $\phi_1$, $\phi_2$ and $\phi_s$ to coalesce smoothly to zero as $A \rightarrow 0$. Figure 6 also shows the limits between which $\theta$ has to lie for solutions to be possible with this choice of $\delta$, and shows how the values have been chosen close to the upper limit to maximise the stable phase-space region. Note the severity of the restrictions for $t > 7.5$msec. Additional considerations are that the bunch at $800MeV$ should have approximately the same length as in the single harmonic model, and that the maximum momentum spread should not exceed previous values.

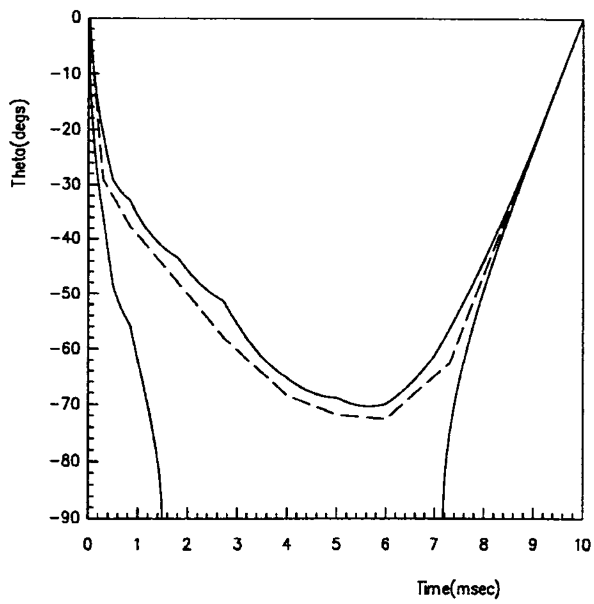


Figure 6. θ-values for ISIS
(The solid curves give the upper and lower limits of θ. The dashed curve shows the values used by the model for ISIS.)

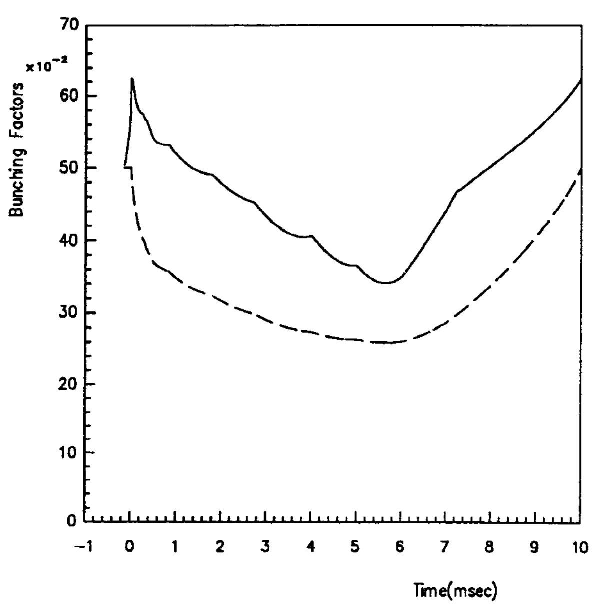


Figure 7. Bunching factors based on a Hofmann-Pedersen Distribution
($\delta \neq 0$: continuous curve; $\delta = 0$: dashed curve)

Bunching factors for this system, based on equation (7), are shown in Figure 7, with a superimposed dashed curve representing values when $\delta = 0$. Throughout the acceleration, the increase is well above 30% and between 0.5msec and 1.5msec is above 45%. In particular, at 1msec, which is the time in ISIS at which space-charge limiting effects are most critical, the bunching factor is increased by 49%. This suggests that it may in fact be possible to accelerate as many as $3.7 \times 10^{13}$ protons per pulse using the dual harmonic system.

The main simulation runs, however, have been carried out for a beam of $3 \times 10^{13}$ protons per pulse, 240$\mu A$ average current. This compares with 200$\mu A$ currently running in the machine. A selection of plots of phase-space is shown in Figure 9, with the superimposed limiting stable region calculated from the Hamiltonian via equation (6). Different colours are used to show the density of charge of the particles, ranging from red at high density through to blue at low density. From top left to bottom right, the plots show the initial formation of a single central core, just after injection but before acceleration and before the dual harmonic waveform is switched on. This develops into two centres as the particles start to orbit around the stable points, $\phi_1$ and $\phi_2$. Beam begins to spiral out and there is some loss (third and fourth pictures) but the 'arms' are largely confined by the separatrix as the bucket contracts. At 6msec, the bucket is expanding again, and as the acceleration slows, as 800$MeV$ is approached, $\phi_1$, $\phi_2$ and $\phi_s$ coalesce and the bunch develops a fairly uniform line density stretching from −65° to 70° (or 120nsec in length). Total loss for the run was calculated to be as low as 3.5% but, whereas for the single harmonic, it was confined to the first 1.5msec, the simulation indicates that for the dual harmonic model, it may continue as far as 6msec into the cycle. One reason for this, as comparison of Figure 8 with Figure 5 suggests, is the reduction in $\delta$ from 0.6 to 0.5 between 3 and 4msec. It may however be possible to avoid the later loss by modifying the voltage profile slightly at earlier times. Such a modification should also reduce the maximum momentum spread, which although generally within $8.5 \times 10^{-3}$ does briefly get as high as $1 \times 10^{-2}$ and is larger than we would ideally like.

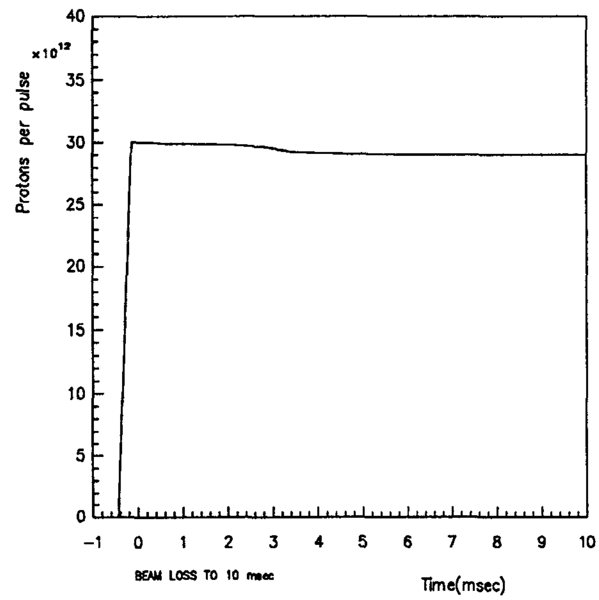


Figure 8. Simulated beam loss for the ISIS model

Initial runs have more recently been carried out for the simulation of $3.7 \times 10^{13}$ protons per pulse. The results have been very encouraging. Loss is slightly greater and again takes place up to 6msec, but is still under 4%. Bunch length at extraction is about 130nsec and maximum momentum spread $8.8 \times 10^{-3}$. These predictions could also probably be improved by slight changes to the cavity voltages early in the cycle.

## *Conclusions*

The study suggests that, by enhancing the bunching factor at critical times and increasing the stable regions of phase-space, use of the dual harmonic waveform should allow more intense beams to be accelerated in synchrotrons. For ISIS the design parameters produced here are within the specifications of the r.f. cavities and it is hoped that some initial experiments may be carried out before the middle of 1994.

| *Time (msec)* | *Peak voltage (kV)* | δ | θ *(degs)* | $\phi_s$ *(degs)* |
|---|---|---|---|---|
| -0.44 | 0.0 | 0.0 | 0.0 | 0.0 |
| -0.35 | 3.0 | 0.0 | 0.0 | 0.0 |
| -0.2 | 6.0 | 0.0 | 0.0 | 0.0 |
| -0.15 | 7.5 | 0.0 | 0.0 | 0.0 |
| 0.0 | 18.3 | 0.5 | 0.0 | 0.0 |
| 0.03 | 20.5 | 0.6 | -2.9 | -5.0 |
| 0.3 | 40.0 | 0.6 | -29.0 | 21.9 |
| 0.5 | 50.0 | 0.6 | -32.2 | 12.2 |
| 0.84 | 75.0 | 0.6 | -37.6 | 17.6 |
| 1.8 | 120.0 | 0.6 | -48.0 | 21.6 |
| 2.72 | 145.0 | 0.6 | -57.9 | 27.9 |
| 4.0 | 160.0 | 0.5 | -68.4 | 36.0 |
| 5.0 | 160.0 | 0.5 | -71.9 | 36.5 |
| 6.0 | 150.0 | 0.5 | -72.6 | 35.6 |
| 7.0 | 145.0 | 0.5 | -64.9 | 37.5 |
| 10.0 | 145.0 | 0.5 | 0.0 | 0.0 |

**Table 1. ISIS Voltage Parameters**

## *References*


1. B W Montague: *Single Particle Dynamics - RF Acceleration* in Preceedings of CERN International School of Particle Accelerators, Erice, 1976.

2. A Hofmann and F Pedersen: *Bunches with Local Elliptic Energy Distributions*, IEEE Trans. Nucl. Sci, Vol. NS-26, No. 3, 1979.

3. G H Rees and J V Trotman: *Acceleration with combined use of R.F. Harmonic Numbers h and 2h*, RAL internal report SCS/MACHINE/10, 1970.

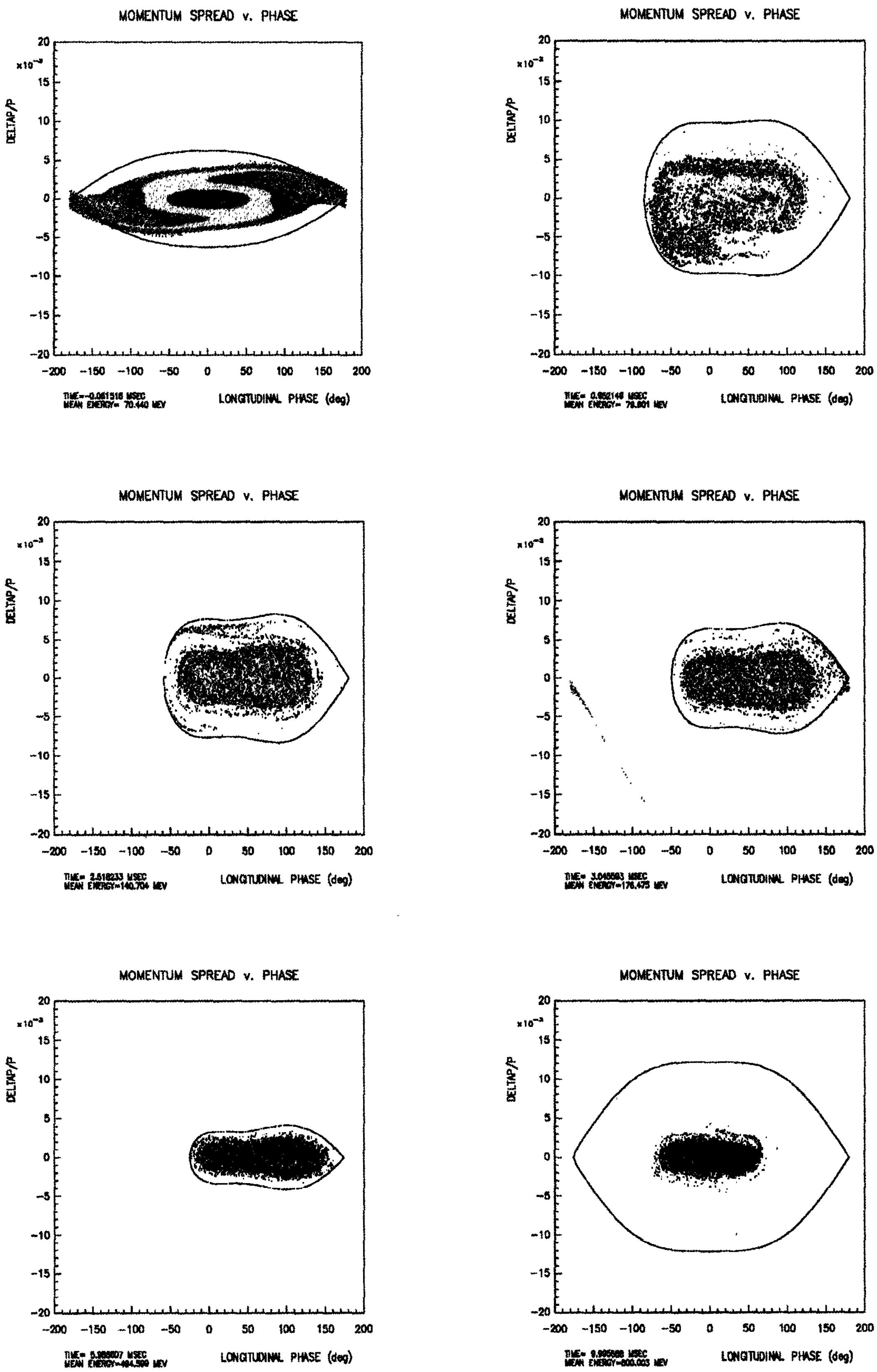


Figure 9. Phase-space plots for the injection and acceleration to $800MeV$ of a beam of $3 \times 10^{13}$ protons per pulse in ISIS.